Theory of Radiative Plasma Focus Model -S Lee Model

[This article is File 2 of a computation package:

There are altogether 4 files in this package.
File1: "Description of Radiative Plasma Focus Computation Code: Click on first http link below:
File2: PDF File "Theory of Radiative Plasma Focus Model" This File
File3: PDF file "Appendix by N A D Khattak".
File4: EXCEL file containing the ACTIVE SHEET AND THE EXECUTABLE (MODIFIABLE) MACRO PROGRAMME CODE.

The latest versions of these files may also be found on the URL:

http://www.kirkbyites.net/DPF/RADPF.htm **or**

http://www.nsse.nie.edu.sg/research/plasmaphysics/ComputationPkg.htm]

This model has been developed for Mather-type (1) plasma focus machines. It was developed for the 3kJ machine known as the UNU/ICTP PFF (2,3) (United Nations University/International Centre for Theoretical Physics Plasma Focus facility, which now forms an international network.In principal there is no limit to energy storage and electrode configuration, though house-keeping may need to be carried out in extreme cases, in order to keep within efficient ranges e.g. of graph plotting.

For details of the computing package, go back to the introductory section.

The model has been used for various applications, for example, in the design of a cascading plasma focus (4); and for estimating soft x-ray yield (5) for the purpose of developing a SXR source for microelectronics lithography (6).

The 5-phase model is described in some detail in the following sections:

1 **Axial Phase**

2 **Radial Inward Shock Phase**

3 **Radial Reflected Shock Phase**

4 **Slow Compression (Radiative) Phase**

5 **Expanded Column Axial Phase**

1 Axial Phase (snow-plow model)

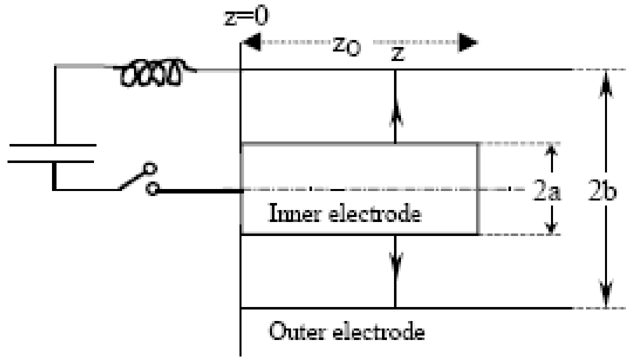

Fig 1 (a) Axial Phase

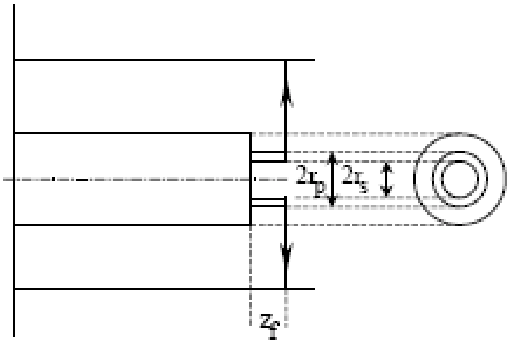

Fig 1 (b) Radial Phase

Rate of change of momentum at current sheath, position z, is

$$\frac{d(mv)}{dt} = \frac{d}{dt}\left(\left[\rho_o \pi (b^2 - a^2) z\right] f_m \frac{dz}{dt}\right) = \rho_o \pi \left(c^2 - 1\right) a^2 \mathrm{f}_m \frac{d}{dt}\left(z \frac{dz}{dt}\right)$$

Magnetic force on current sheath is

$$F = \int_a^b \left[\left(\frac{\mu I f_c}{2\pi r}\right)^2 /(2\mu)\right] 2\pi r dr = \frac{\mu \mathrm{f}_c^{\,2}}{4\pi} \ln(c) I^2$$

f_m = fraction of mass swept down the tube in the axial direction
f_c = fraction of current flowing in piston

Equation of motion:

$$\rho_o \pi (c^2 - 1) a^2 \mathrm{f_m} \frac{d}{dt}\left(z \frac{dz}{dt}\right) = \frac{\mu \mathrm{fc}^2}{4\pi} (\ln c) I^2$$

$$\therefore \frac{d^2 z}{dt^2} = \left[\frac{\mathrm{f_c}^2}{f_m} \frac{\mu (\ln c)}{4\pi^2 \rho_o (c^2 - 1)} \left(\frac{I}{a}\right)^2 - \left(\frac{dz}{dt}\right)^2\right] / z \qquad \text{-- (I)}$$

Circuit (current) Equation

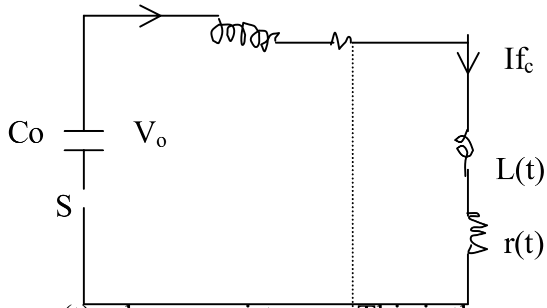

Fig 2 Circuit schematic

Ignore r(t), plasma resistance. This is the approximation generally used for electromagnetic drive.

$$\frac{d}{dt}\left[(L_o + L\mathrm{f_c})I\right] + r_o I = V_o - \int \frac{Idt}{C_o}$$

$$(L_o + L\mathrm{f_c})\frac{dI}{dt} + I\mathrm{f_c}\frac{dL}{dt} + r_o I = V_o - \int \frac{Idt}{C_o} \qquad \text{-- (II)}$$

$$\frac{dI}{dt} = \left[V_o - \frac{\int Idt}{C_o} - r_o I - If_c \frac{\mu}{2\pi}(\ln c)\frac{dz}{dt}\right] / \left[L_o + \frac{f_c \mu}{2\pi}(\ln c) z\right]$$

Equations (I) and II) are the generating equations of the model.
They contain the physics built into the model.

They are coupled equations.

The equation of motion is affected by the electric current I.

The circuit equation is affected by the current sheath motion dz/dt and position z.

Normalise the equations to obtain scaling parameters

Replace variables t, z, I by non-dimensionalised quantities as follows:

$$\tau = t/t_o, \quad \varsigma = z/z_o, \quad \iota = I/I_o$$

where the normalising quantities t_o, I_o and Z_o are carefully chosen to be relevant, <u>characteristic</u>, convenient quantities, reflecting the physics of the problem.

Choices:

z_o is the length of the anode,

t_o is $\sqrt{L_o C_o}$ (noting that $2\pi\sqrt{L_o C_o}$ is the cycle time of L_o-C_o discharge circuit)

I_o is V_o/Z_o where $Z_o = \sqrt{L_o/C_o}$ is the surge impedance (noting that I_o is the peak current of the L_o-C_o discharge circuit with capacitor C_o charged initially to V_o.)
Normalising, we have:

Equation of motion:

$$\frac{z_o}{t_o^2}\frac{d^2\zeta}{d\tau^2} = \left[\frac{\mathrm{f}_c^2}{\mathrm{f_m}}\frac{\mu \ln c}{4\pi^2 \rho_o (c^2-1)}\left(\frac{I_o}{a}\right)^2 \iota^2 - \frac{z_o{}^2}{t_o^2}\left(\frac{d\zeta}{d\tau}\right)^2\right] \Big/ \zeta z_o$$

$$\frac{d^2\zeta}{d\tau^2} = \left[\frac{\mathrm{f_c}^2}{\mathrm{f_m}}\frac{\mu \ln c}{4\pi^2 \rho_o (c^2-1)}\left(\frac{I_o}{a}\right)^2 \frac{t_o^2}{z_o^2}\iota^2 - \left(\frac{d\zeta}{d\tau}\right)^2\right] \Big/ \zeta$$

which we write as

$$\frac{d^2\zeta}{d\tau^2} = \frac{\left[\alpha^2 \iota^2 - \left(\frac{d\zeta}{d\tau}\right)^2\right]}{\zeta} \qquad \text{-- (I.1)}$$

Obtain first scaling parameter:

We note, by inspection,

$\alpha^2 = t_o^2 / \left\{ \left[z_o^2 / (I_o / a)^2 \right] \left(\mathrm{f_m} / \mathrm{f_c^2} \right) \left[4\pi^2 \rho_o \left(c^2 - 1 \right) / \mu \ln c \ \right] \right\}$, which we thus define in this manner.

By inspection of equation (I.1), we note α is dimensionless.

Hence since t_o has the dimension of time we may define a time value t_a where

$$t_a = \left[\frac{4\pi^2 \left(c^2 - 1\right)}{\mu \ln c} \right]^{1/2} \frac{\sqrt{\mathrm{f_m}}}{\mathrm{f_c}} \frac{z_o}{\left(I_o / a\right) / \sqrt{\rho}}$$

identifying this quantity as the characteristic axial transit time of the CS down the anode axial phase.

We may then think of α as:

$\alpha = (t_o/t_a)$ – scaling parameter.

ratio of characteristic electrical discharge time to characteristic axial transit time.

We may further identify a characteristic axial transit speed $V_a = z_o/t_a$

$$v_a = \left[\frac{\mu \ln c}{4\pi^2 \left(c^2 - 1\right)} \right]^{1/2} \frac{\mathrm{f_c}}{\sqrt{\mathrm{f_m}}} \frac{\left(I_o / a\right)}{\sqrt{\rho}}$$

The quantity $\left(I_o/a \right) / \sqrt{\rho}$ is the S (speed or drive) factor of electromagnetically driven devices, focus, pinches etc.

Normalising the circuit (current) Equation, we have:

$$\frac{I_o}{t_o}\frac{d\iota}{d\tau} = \left[v_o - \frac{I_o t_o}{c_o} \int \iota d\tau - r_o I_o \iota - f_c \frac{\mu}{2\pi 1} (\ln c) I_o \frac{z_o}{t_o} \iota \frac{d\zeta}{d\tau} \right] \Big/ \left[L_o + \frac{\mathrm{f}_c \mu}{2\pi} (\ln c) z_o \zeta \right]$$

and substituting in $I_o = V_o / \sqrt{L_o / C_o}$, $t_o = \sqrt{L_o C_o}$, we have

$$\frac{d\iota}{d\tau} = \left[1 - \int \iota d\tau - \mathrm{f_c} \left[\left(\frac{\mu}{2\pi} (\ln c) z_o \right) / L_o - (r_o / Z_{o)} \right] \iota \frac{d\zeta}{d\tau} \right] \left[1 + \mathrm{f_c} \left[\frac{\mu}{2\pi} (\ln c) z_o \right] \zeta / L_o \right]$$

write: $\frac{d\iota}{d\tau} = \left(1 - \int \iota d\tau - \beta \iota \frac{d\zeta}{d\tau} - \delta \iota \right) / \left(1 + \beta \zeta \right)$ -- (II.1)

Second scaling parameter

We note $L_a = \frac{f_c \mu}{2\pi}(\ln c) z_o$ is the inductance of the axial phase when CS reaches the end $z = z_o$.

Hence $\beta = \frac{L_a}{L_o}$ is the ratio of load to source inductance and since the device is electromagnetic, the electrodynamics is determined strongly by this scaling parameter.

The third scaling parameter $\delta = r_o / Z_o$ is the ratio of circuit stray resistance to surge impedance. This acts as a damping effect on the current.

(I.1) and (II.1) are the Generating Equations that may be integrated step-by-step.

Calculate voltage across input terminals of focus tube:

$$V = \frac{d}{dt}(LIf_c) = f_c I \frac{dL}{dt} + f_c L \frac{dI}{dt} \quad where \quad L = \frac{\mu}{2\pi}(l_n c) \qquad \text{- (II.10)}$$

Normalized form $\upsilon = \frac{V}{V_o} = \beta\iota\frac{d\zeta}{d\tau} + \beta\zeta\frac{d_c}{d\tau}$ - (II.11)

Integration

Define initial conditions:

$$\tau = 0, \quad \frac{d\zeta}{d\tau} = 0, \quad \zeta = 0, \quad \iota = 0, \quad \int \iota d\tau = 0, \quad \frac{d\iota}{d\tau} = 1, \quad \frac{d^2\zeta}{d\tau^2} = \alpha\sqrt{2/3}$$

Set time increment: D = 0.001

Increment time: $\tau = \tau + D$

Next step values are computed using the following linear approximations:

$$\frac{d\zeta}{d\tau} = \frac{d\zeta}{d\tau} + \frac{d^2\zeta}{d\tau^2} D$$
$$\zeta = \zeta + \frac{d\zeta}{d\tau} D$$
$$\iota = \iota + \frac{d\iota}{d\tau} D$$
$$\int \iota d\tau = \int \iota d\tau + \iota D$$

Use new values of $\frac{d\zeta}{d\tau}, \zeta, \iota \; and \int \iota d\tau$ to calculate new generating values of $\frac{d\iota}{d\tau}$ and $\frac{d^2\zeta}{d\tau^2}$ using generating equs (I.1) and (II.1).

Increment time again and repeat calculations of next step values and new generating values.

Continue procedure until $\zeta = 1$.

Then go on to radial phase inward shock computations.

2 Radial Inward ShockPhase (Slug model)

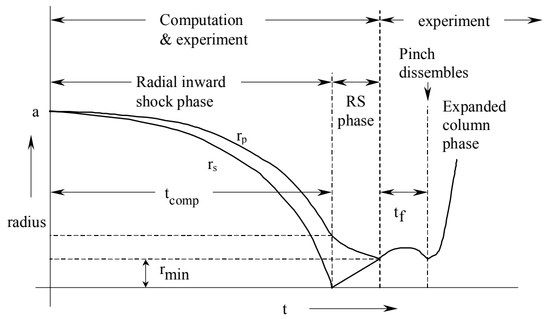

Fig 3. Schematic of radial phases

The snowplow model is used for axial phase just to obtain axial trajectory and speed (from which temperature may be deduced) and to obtain reasonable current profile. As the CS is assumed to be infinitesmally thin, no information of density is contained in the physics of the equation of motion, although an estimate of density may be obtained by invoking additional mechanisms e.g. using shock wave theory.

In the radial phase however, a snowplow model (with an infinitesmally thin CS) would eventually (in the integration) lead to all current flowing at r = 0, with infinite inductance and density.

We thus replace the snow plow model by a slug model . In this model, the magnetic pressure drives a shock wave ahead of it, creating a space for the magnetic piston (CS) to move into.

The speed of the inward radial shock front (see Fig 1b)is determined by the magnetic pressure (which depends on the drive current value and CS position r_p).

The speed of the magnetic piston (CS) is determined by the first law of thermodynamics applied to the effective increase in volume between SF and CS, created by the incremental motion of the SF.

The compression is treated as an elongating pinch.

Four generating equations are needed to describe the motion of (a) SF, (b) CS (c) pinch elongation and the electric current (d); to integrate for the four variables r_s, r_p, z_f & I.

Motion of Shock Front:

r_p piston r_s SF

P_m vacuum | P ρ | ρ_o, P_o, T_o

T

From Shock Wave theory, shock pressure $P = \frac{2}{\gamma+1}\rho_o v_s^{\,2}$ where shock speed v_s into ambient gas ρ_o causes the pressure of the shocked gas (just behind the shock front) to rise to value P.

If we assume that this pressure is uniform from the SF to the CS (infinite acoustic [small disturbance] speed approximation) then across the piston, we may apply P = P_m where

$$P_m = \left(\mu I f_c / 2\pi r_p\right)^2 / 2\mu$$

Thus: $$v_s^2 = \frac{\mu (I \mathrm{f_c})^2}{8\pi^2 r_p} x \frac{\gamma+1}{2\rho_o \mathrm{f_{mr}}}$$

where I is the circuit current and Ifc is the current flowing in the cylindrical CS, taken as the same f_c as in the axial phase, and $\rho_o f_{mr}$ is the effective mass density swept into the radial slug; where f_{mr} is a different (generally larger) factor than f_m of the axial phase.

Thus $$\frac{dr_s}{dt} = -\left[\frac{\mu(\gamma+1)}{\rho_o}\right]^{1/2} \frac{\mathrm{f_c}}{\sqrt{\mathrm{f_{mr}}}} \frac{I}{4\pi r_p} \qquad \text{-- (III)}$$

Elongation speed of CS (open-ended at both ends)

The radial compression is open at one end. Hence an axial shock is propagated in the z-direction, towards the downstream anode axis. We take z_f as the position of the axial CS (rather than the SF). The pressure driving the axial shock is the same as the pressure driving the inward radial shock. Thus the axial shock speed is the same as the radial shock speed. The CS speed is slower, from shock wave theory, by an approximate factor of $2/(\gamma+1)$. Thus the axial elongation speed of the CS is:

$$\frac{dz_f}{dt} = -\left(\frac{2}{\gamma+1}\right)\frac{dr_s}{dt} \qquad \text{-- (IV)}$$

In this modelling we treat the elongation in a very approximate fashion, as its effect on the compressing column is relatively secondary. The main mechanism controlling the state of the plasma column is the radial compression. The radial CS (piston) speed is hence treated with more care as follows:

Radial piston motion

We inquire:

For an incremental motion, dr_s, of the shock front, at a driving current I, what is the relationship between plasma slug pressure P and plasma slug volume V?

We assume an adiabatic relationship (7) (infinite small disturbance speed – for which we will apply a correction subsequently) to a fixed mass of gas in the slug during the incremental motion dr_s. We have

PV^{γ} = constant or

$$\frac{\gamma dV}{v} + \frac{dP}{P} = 0$$

where slug pressure $P \sim v_s^2$

so $\frac{dP}{P} = \frac{2dv_s}{v_s}$

but $v_s \sim \frac{I}{r_p}$ (see section on Motion of Shock Front, above)

so $\frac{dP}{P} = 2\left(\frac{dI}{I} - \frac{dr_p}{r_p}\right)$

Now slug volume $V = \pi\,(r_p^2 - r_s^2)\, z_f$

and at first sight $dV = 2\pi\,(r_p dr_p - r_s\, dr_s)\, z_f + \pi\,(r_p^2 - r_s^2)\, dz_f$ – not correct!

But here we note that although the motion of the piston dr_p does not change the mass of gas in the slug, the motion of the shock front, dr_s, does sweep in an amount of ambient gas.

This amount swept in is equal to the ambient gas swept through by the shock front in its motion dr_s. This swept-up gas is compressed by a ratio $(\gamma+1)/(\gamma-1)$ and will occupy part of the increase in volume dV.
The actual increase in volume available to the original mass of gas in volume V does not correspond to increment dr_s but to an effective (reduced) increment drs $(2/(\gamma+1))$. (Note γ is specific heat ratio of the plasma e.g. $\gamma = 5/3$ for atomic gas, $\gamma = 7/5$ for molecular gas; for strongly ionising argon γ has value closer to 1 e.g. 1.15.)

Thus, the more correct interpretation is:

$$dV = 2\pi\left(r_p dr_p - \frac{2}{\gamma+1} r_s dr_s\right) z_f + \pi\left(r_p^2 - r_s^2\right) dz_f$$

Thus we have: $$\frac{\gamma dV}{V} = \frac{2\gamma\left(r_p dr_p - \frac{2}{\gamma+1} r_s dr_s\right) z_f + \gamma\left(r_p^2 - r_s^2\right) dz_f}{z_f\left(r_p^2 - r_s^2\right)}$$

and adding together dP/P and γdV/V we have

$$\frac{2\gamma\left(r_p dr_p - \frac{2}{\gamma+1} r_s dr_s\right) z_f + \gamma\left(r_p^{\,2} - r_s^{\,2}\right) dz_f}{z_f\left(r_p^{\,2} - r_s^{\,s}\right)} + 2\frac{dI}{I} - \frac{2dr_p}{r_p} = 0$$

Rearranging and putting dr_p as the subject we have

$$\frac{dr_p}{dt} = \frac{\frac{2}{\gamma+1}\frac{r_s}{r_p}\frac{dr_s}{dt} - \frac{r_p}{\gamma I}\left(1 - \frac{r_s^{\,2}}{r_p^{\,2}}\right)\frac{dI}{dt} - \frac{r_p}{z_f}\left(1 - \frac{r_s^{\,2}}{r_p^{\,2}}\right)\frac{dz_f}{dt}}{\frac{\gamma-1}{\gamma} + \frac{1}{\gamma}\frac{r_s^{\,2}}{r_p^{\,2}}} \qquad \text{-- (V)}$$

where we are reminded r_p = radial piston position
r_s = radial shock front position
z_f = axial piston position

Circuit Equation during radial phase

The inductance of the focus tube now consists of the full inductance of the axial phase and the inductance of the radially imploding & elongating plasma pinch.

Thus $L = \frac{\mu}{2\pi}(\ln c) z_o + \frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right) z_f$ where both z_f and r_p vary with time.

Thus the circuit (current) equation is

$$\left\{L_o + f_c\frac{\mu}{2\pi}(\ln c) z_o + f_c\frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right) z_f\right\}\frac{dI}{dt} + f_c I\frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right)\frac{dz_f}{dt} - f_c I\frac{\mu}{2\pi}\frac{z_f}{r_p}\frac{dr_p}{dt} + r_o I$$
$$= V_o - \frac{\int I dt}{C_o}$$

Giving
$$\frac{dI}{dt} = \frac{V_o - \frac{\int I dt}{C_o} - r_o I - f_c\frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right) I\frac{dz_f}{dt} + f_c\frac{\mu}{2\pi}\frac{z_f}{r_p} I\frac{dr_p}{dt}}{L_o + f_c\frac{\mu}{2\pi}(\ln c) z_o + f_c\frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right) z_f} \qquad \text{-- (VI)}$$

Generating equations (III), (IV), (V), (VI) form a close set of equations which may be integrated for r_s, r_p, z_f and I.

Normalization

For this phase the following normalization is adopted.

$\tau = t/t_o$, $\iota = I/I_o$ as in axial phase but with $\kappa_s = r_s/a$, $\kappa_p = r_p/a$, $\zeta_f = z_f/a$ ie. distances are normalized to anode radius, instead of anode length.

After normalization we have:

Radial shock speed $$\frac{d\kappa_s}{d\tau} = -\alpha\alpha_1 \iota / \kappa_p \qquad \text{-- (III.1)}$$

Axial column elongation speed (both ends of column defined by axial piston)

$$\frac{d\zeta_f}{d\tau} = -\frac{2}{\gamma+1}\frac{d\kappa_s}{d\tau} \qquad \text{-- (IV.1)}$$

Radial piston speed: $$\frac{d\kappa_p}{d\tau} = \frac{\frac{2}{\gamma+1}\frac{\kappa_s}{\kappa_p}\frac{d\kappa_s}{d\tau} - \frac{\kappa_p}{\gamma\iota}\left(1-\frac{\kappa_s^2}{\kappa_p^2}\right)\frac{d\iota}{d\tau} - \frac{1}{\gamma+1}\frac{\kappa_p}{\zeta_f}\left(1-\frac{\kappa_s^2}{\kappa_p^2}\right)\frac{d\zeta_f}{d\tau}}{(\gamma-1)/\gamma + (1/\gamma)(\kappa_s^2/\kappa_p^2)} \qquad \text{-- (V.1)}$$

current: $$\frac{d\iota}{d\tau} = \frac{1 - \int \iota d\tau + \beta_1\left[\ln(\kappa_p/c)\right]\iota\frac{d\zeta_f}{d\tau} + \beta_1\frac{\zeta_f \iota}{k_p}\frac{d\kappa_p}{d\tau} - \delta\iota}{\left\{1+\beta-(\beta_1)\left[\ln\left(\kappa_p/c\right)\right]\zeta_f\right\}} \qquad \text{(VI.1)}$$

where the **scaling parameters** are $\beta_1 = \beta/(F\ln c)$, $F = z_o/a$ and
$\alpha_1 = \left[(\gamma+1)\left(c^2-1\right)/(4\ln c)\right]^{1/2} F\left[f_m/f_{mr}\right]^{1/2}$

Note that whereas we interpret $\alpha = t_o/t_a$, we may interpret $\alpha_1 = t_a/t_r$ where t_r is a characteristic radial transit time.

The **scaling parameter** $\alpha\alpha_1$ may then be interpreted as $\alpha\alpha_1 = \frac{t_o}{t_a} = t_o/t_r$

We note that α_1 the ratio of characteristic axial transit to characteristic radial compression inward shock transit time is essentially a geometrical ratio

$F\left[\left(c^2-1\right)/4\ln c\right]^{1/2} \simeq 20$ (*if* F ≃ 16 and c ≃ 3)
(ie. axial transit time is characteristically 20 times longer than radial shock transit) modified by the thermodynamic term $(\gamma+1)^{1/2}$ and the mass swept up ratio $\left(f_m/f_{mr}\right)^{1/2}$. Including all 3 factors, the ratio of axial to radial characteristic times is typically 40.

We also note from the scaling parameter $\alpha\alpha_1$ that

$$t_r = \frac{4\pi}{\left[\mu(\gamma+1)\right]^{1/2}}\frac{\sqrt{f_{mr}}}{f_c}\frac{a}{(I_0/a/\sqrt{\rho}}$$

and characteristic speed of inward shock to reach focus axis is:

$$v_r = a/t_r = \frac{[\mu(\gamma+1)]^{1/2}}{4\pi} \frac{f_c}{\sqrt{f_{mr}}} \frac{(I_o/a)}{\sqrt{\rho}}$$

The ratio of characteristic radial and axial speeds is also essentially a geometrical one, modified by thermodynamics. It is $v_r/v_a = \left[\frac{(c^2-1)(\gamma+1)}{4\ln c}\right]^{1/2}$ with a value typically 2.5.

Note that the radial characteristic speed has the same dependence as the axial transit speed on drive factor $S = (I_o/a)/\sqrt{\rho}$.

Note on Time Match Safeguard:

Time match ratio denoted as ALT in the programme code.

$$ALT = t_0/(t_a+t_r) = \alpha\alpha_1/(\alpha_1+1)$$

Calculate voltage V across PF input terminals

As in the axial phase, the voltage is taken to have only inductive component.

$$V = \frac{d}{dt}(LI)$$

where $L = \frac{\mu}{2\pi}(\ln c)z_o + \frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right)z_f$

$$V = \frac{\mu}{2\pi}\left[(\ln c)z_o + \left(\ln\frac{b}{r_p}\right)z_f\right] f_c \frac{dI}{dt} + \frac{\mu}{2\pi}\left[\left(\ln\frac{b}{r_p}\right)\frac{dz_f}{dt} - \frac{z_f}{r_p}\frac{dr_p}{dt}\right] f_c I \qquad \text{- (VI.10)}$$

We may also write in normalised form $\upsilon = V/V_o$
(normalised to initial capacitor voltage V_o)

$$\nu = \left[\beta - \beta_1\left(\ln\frac{\kappa_p}{c}\right)\zeta_f\right]\frac{d\iota}{d\tau} - \beta_1\iota\left[\left(\frac{\zeta_f}{\kappa_p}\right)\frac{d\kappa_p}{d\tau} + \left(\ln\frac{\kappa_p}{c}\right)\frac{d\zeta_f}{d\tau}\right] \qquad \text{- (VI.12)}$$

The generating equations (III.1), (IV.1), (V.1), (VI.1) may now be integrated using the following initial conditions:

τ = the time that axial phase ended
$\kappa_s = 1$
$\kappa_p = 1$
$\zeta_f = 0$ (taken as a small number such as 0.00001 to avoid numerical difficulties for equation V.1)
ι = value of current at the end of the axial phase.
$\int \iota d\tau$ = value of 'flowed charge' at end of axial phase.

The integration (step-by-step) may proceed with the following algorithm: (taking smaller time increment of D = 0.001/100)
Using initial values (above) of κ_s, κ_p, ζ_f and ι

$\frac{d\kappa_s}{d\tau}, \frac{d\zeta_f}{d\tau}, \frac{d\kappa_p}{d\tau}$ *and* $\frac{d\iota}{d\tau}$ are sequentially calculated from generating equation (III.1), (IV.1), (V.1), (VI.1).

Then sequentially using linear approximation:

$$\kappa_s = \kappa_s + \frac{d\kappa_s}{d\tau} D$$

$$\zeta_f = \zeta_f + \frac{d\zeta_f}{d\tau} D$$

$$\kappa_p = \kappa_p + \frac{d\kappa_p}{d\tau} D$$

$$\iota = \iota + \frac{d\iota}{d\tau} D$$

$$\int \iota d\tau = \int \iota d\tau + \iota D$$

Time is then incremented by D, and the next step value of $\frac{d\kappa_s}{d\tau}, \frac{d\zeta_f}{d\tau}, \frac{d\kappa_p}{d\tau}, \frac{d\iota}{d\tau}$ are computed from (III.1), (IV.1), (V.1) and (VI.1), followed by linear approximation for κ_s, ζ_f, κ_ρ, ι and $\int \iota d\tau$.

The sequence is repeated step-by-step until $\kappa_s = 0$.

Correction for finite acoustic (small disturbance) speed.

In the slug model above we assume that the pressure exerted by the magnetic piston (current I, position r_p) is instantaneously felt by the shock front (position r_s). Likewise the shock speed $\frac{dr_s}{dt}$ is instantaneously felt by the piston (CS). This assumption of inifinite small disturbance speed (SDS) is implicit in equations (III) and (V) (or in normalised form (III.1) and (V.1)).

Since the SDS is finite, there is actually a time lapse Δt communicating between the SF and CS. This communication delay has to be incorporated into the model. Otherwise for the PF, the computation will yield too high values of CS and SF speed.

Consider the instant t, SF is at r_s, CS at r_p, value of current is I. SF actually feels the effect of the current not of value I but of a value I_{delay} which flowed at time (t-Δt), with the CS at r_{pdelay}. Similarly the piston 'thinks' the SF speed is not $\frac{dr_s}{dt}$ but $\left(\frac{dr_s}{dt}\right)_{delay}$ at time (t-Δt).

To implement this finite SDS correction we adopt the following procedure:

Calculate the SDS, taken as the acoustic speed.

$$SDS = \left(\frac{\gamma P}{\rho}\right)^{1/2} \; or \left(\frac{\gamma R_o}{M} D_c T\right)^{1/2}$$

$$or \quad \left(\frac{\gamma D_c kT}{M m_i}\right)^{1/2}$$

where γ = specific heat ratio, M = Molecule Weight,
R_o = universal Gas constant = 8 x 10^3 (SI units)
m_i = mass of proton, k = Boltzmanns constant.
D_c = departure coefficient = DN (1+z)
where Z, here, is the effective charge of the plasma

$Z = \sum_r^J r\alpha_r$, summed over all ionization levels r = 1… J.

DN = dissociation number, e.g. for Deuterium DN = 2, whereas for argon DN = 1.

The temperature T may be computed for the shocked plasma as

$$T = \frac{M}{RoD} \frac{2(\gamma-1)}{(\gamma+1)^2} \left(\frac{dr_s}{dt} \right)^2$$

Calculate the communication delay time as
$\Delta T = (r_p - r_s)/SDS$

In our programme using the Microsoft EXCEL VISUAL BASIC, data of the step-by-step integration is stored row-by-row, each step corresponding to one row. Thus the ΔT may be converted to Δ (row number) by using Δ (row number) = ΔT/(timestep increment) this Δ (row number being, of course, rounded off to an integer.

The correction then involves 'looking back' to the relevant row number to extract the corrected values of I_{delay}, r_{pdelay}, $\left(\frac{dr_s}{dt} \right)_{delay}$.

Thus in the actual numerical integration, in equation (III.1), ι and κ_p are replaced by ι_{delay} and κ_{pdelay} and in equation (V.1) $\frac{dk_s}{d\tau}$ is replaced by $\left(\frac{dk_s}{d\tau} \right)_{delay}$

Radial Reflected Shock Phase

When the inward radial shock hits the axis, $\kappa_s = 0$. Thus in the computation, when $\kappa_s \leq 1$ we exit from radial inward shock phase. We start computing the RS phase.

The RS is given a constant speed of 0.3 of on-axis inward radial shock speed.

In this phase computation is carried out in real (SI) units.

Reflected Shock Speed:

$$\frac{dr_r}{dt} = -0.3 \left(\frac{dr_s}{dt} \right)_{on-axis}$$

Piston speed:

$$\frac{dr_p}{dt}=\frac{-\frac{r_p}{\gamma I}\left(1-\frac{r_s^{\,2}}{r_p^{\,2}}\right)\frac{dI}{dt}-\frac{r_p}{z_{\mathrm{f}}}\left(1-\frac{r_s^{\,2}}{r_p^{\,2}}\right)\frac{dz_{\mathrm{f}}}{\mathrm{dt}}}{\frac{\gamma-1}{\gamma}+\frac{1}{\gamma}\frac{r_s^{\,2}}{r_p^{\,2}}}$$

Use the same equation as V except put $\left(\frac{dr_s}{dt}\right)=0$

Elongation speed:

Use same equation as Eq IV.

$$\frac{dz_f}{dt}=-\left(\frac{2}{\gamma+1}\right)\left(\frac{dr_s}{dt}\right)_{on-axis}$$

Circuit Equation:

Use the same equation as Eq VI.

$$\frac{dI}{dt}=\frac{V_o-\frac{\int Idt}{C_o}-r_oI-\mathrm{f_c}\frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right)I\frac{dz_{\mathrm{f}}}{dt}+\mathrm{f_c}\frac{\mu}{2\pi}\frac{z_{\mathrm{f}}}{\mathrm{r_p}}I\frac{dr_p}{dt}}{L_o+\mathrm{f_c}\frac{\mu}{2\pi}(\ln c)z_o+\mathrm{f_c}\frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right)z_{\mathrm{f}}}$$

Continue integrating seamlessly.

Tube Voltage

Use the same equation as Eq (VI.10).

$$V=\frac{\mu}{2\pi}\left[(\ln c)z_o+\left(\ln\frac{b}{r_p}\right)z_{\mathrm{f}}\right]\mathrm{f_c}\frac{dI}{dt}+\frac{\mu}{2\pi}\left[\left(\ln\frac{b}{r_p}\right)\frac{dz_{\mathrm{f}}}{\mathrm{dt}}-\frac{z_{\mathrm{f}}}{\mathrm{r_p}}\frac{dr_p}{dt}\right]\mathrm{f_c}I$$

In this phase as the RS (position r_r) moves outwards, the piston (position r_p) continues moving inwards.

Eventually r_r increases until its value reaches the decreasing value of r_p.

We make the assumption that the RS is sufficiently attenuated when it reaches the piston, so that its overpressure is negligible.

In that case, the piston may not be pushed outwards, but will continue to move inwards, although its inward speed may be gradually reduced.

4 Slow Compression Phase

In this phase the piston speed is:

$$\frac{dr_p}{dt}=\frac{\dfrac{-r_p}{\gamma I}\dfrac{dI}{dt}-\dfrac{1}{\gamma+1}\dfrac{r_p}{z_f}\dfrac{dz_f}{dt}+\dfrac{4\pi(\gamma-1)}{\mu\gamma z_f}\dfrac{r_p}{f_c^2 I^2}\dfrac{dQ}{dt}}{\dfrac{\gamma-1}{\gamma}} \qquad \text{- (XX)}$$

Here we have included energy loss/gain terms into the equation of motion. The plasma gains energy from Joule heating. Using Spitzer form for resistivity we have for the plasma column.

$$\frac{dQ_J}{dt}=RI^2 f_c^2 \quad where \quad R=\frac{1290Zz_f}{\pi r_p^2 T^{3/2}}$$

$$T=\frac{\mu}{8\pi^2 k}I^2 f_c^2/(DN_o a^2(\text{fm}))$$

The Bremstrahlung loss term may be written as:

$$\frac{dQ_B}{dt}=-1.6x10^{-40}N_i^2(\pi r_p^2)z_f T^{1/2}z^3$$

$$N_o=6x10^{26}\frac{\rho_o}{M}; \quad N_i=N_o f_{mr}\left(\frac{a}{r_p}\right)^2$$

The line loss term may be written as:

$$\frac{dQ_L}{dt}=-4.6x10^{-31}N_i^2 ZZ_n^4(\pi r_p^2)z_f/T$$

$$and\ \frac{dQ}{dt}=\frac{dQ_J}{dt}+\frac{dQ_B}{dt}+\frac{dQ_L}{dt}$$

By this coupling, if, for example, the radiation loss $\left(\frac{dQ_B}{dt}+\frac{dQ_L}{dt}\right)$ is severe, this would lead to a large value of $\frac{dr_p}{dt}$ inwards. In the extreme case, this leads to radiation collapse, with r_p going rapidly to zero, or to such small values that the plasma becomes opaque to the outgoing radiation, thus stopping the radiation loss.

This radiation collapse occurs at a critical current of 1.6 MA (the Pease-Braginski current) for deuterium. For gases such as Neon or Argon, because of intense line radiation, the critical current is reduced to even below 100kA, depending on the plasma temperature.

Column elongation

Whereas in the radial RS phase we have adopted a 'frozen' elongation speed model, we now allow the elongation to be driven fully by the plasma pressure.

$$\frac{dz_f}{dt} = \left[\frac{\mu}{4\pi^2(\gamma+1)\rho_o} \right]^{1/2} \frac{If_c}{r_p} \qquad \text{- (XXI)}$$

Circuit current equation

$$\frac{dI}{dt} = \frac{V_o - \frac{\int Idt}{C_o} - \frac{\mu}{2\pi}\left(\ln\frac{b}{r_p}\right)\frac{dz_f}{dt} If_c + \frac{\mu}{2\pi}\frac{z_f}{r_p}\frac{dr_p}{dt} If_c - I(Rf_c + r_o)}{L_o + \frac{\mu}{2\pi} f_c \left((\ln c) z_o + \left(\ln\frac{b}{r_p}\right) z_f \right)} \qquad \text{- (XXII)}$$

Equations (XX), (XXI) and (XXII) are integrated as coupled equations for r_p, z_f and I. At each step the value of $\frac{dQ}{dt}$ is also evaluated as above.

The total energy radiated by Bremsstrahlung (Q_B) and line radiation (Q_L) may also be evaluated.

Voltage across focus terminals

$$V = \frac{\mu f_c}{2\pi} I \left[\left(\ln\frac{b}{r_p}\right)\frac{dz_f}{dt} - \frac{z_f}{r_p}\frac{dr_p}{dt} \right] + \frac{\mu f_c}{2\pi}\left[\left(\ln\frac{b}{r_p}\right) z_f + (\ln C) z_o \right] \frac{dI}{dt} + RI$$

Instability resistance/impedance not included in slow compression phase

From experiments, it is well known that after a brief period (few ns), the quiescent column is rapidly broken up by instabilities. One effect is a huge spike of voltage, partially observed at focus tube terminals. This voltage spike is responsible for driving ion beams (forward direction) and REB (negative direction, up the anode) with energies typically 200keV.

We could model this by including a suitable time varying resistance/impedance into the $\frac{dI}{dt}$ equation; and adjusting this function to suit the observed voltage/beam energy characteristics. There is a complication of this ‘annomalous’ resistance in our modelling. If we include this resistance also into the joule heating term in the piston motion Eq (XX), the sudden increase in $\frac{dQ_J}{dt}$ will blow the piston outwards, leading to a huge negative voltage ‘spike’; not experienced experimentally. The model may be more realistic if at the moment of introducing the ‘annomalous’ resistance, the piston motion is ‘frozen’, or even allowed to continue inwards, as the magnetic field in such ‘small Magnetic Reynolds Number’ situation will diffuse inwards – no piston blow-out!

The final results of this instability mechanism is the breaking up of the focus pinch into a large expanded current column.

5 Expanded Column Axial Phase

We model the expanded column axial phase (3,4) in the following manner.

In the expanded column phase we assume that the current flows uniformly from anode to cathode in a uniform column having the same radius as the anode and a length of z.

The normalised equations (same normalisation as in axial phase):

Circuit current: $$\frac{d\iota}{d\tau} = \frac{1 - \int \iota d\tau - \beta\iota \frac{d\zeta}{d\tau} e - \delta\iota}{1 + \beta + \beta(\zeta - 1)e}$$

where $e = \left(\ln c + \frac{1}{z} \right) / \ln c$

Motion: $$\frac{d^2\zeta}{d\tau^2} = \frac{\alpha^2 \iota^2 e_1 - h^2 \left(\frac{d\zeta}{d\tau} \right)^2}{1 + h^2(\zeta - 1)}$$

where
$$h = \left[c^2 / (c^2 - 1) \right]^{1/2}$$
$$e_1 = \left(\ln c + \frac{1}{4} \right) / (\ln c)$$

The initial conditions for ι and $\int \iota d\tau$ are the last values of ι and $\int \iota d\tau$ from the last phase. The initial value of ζ is $\zeta = 1 + \zeta_f$ where ζ_f is the last length of the focus column, but normalised to z_o, rather than a.

References

1 J W Mather. Phys Fluids, **8**, 366 (1965)

2 S Lee, T Y Tou, S P Moo, M A Eissa, A V Gholap, K H Kwek, S Mulyodrono, A J Smith, Suryadi, W Usada, M Zakaullah. Amer J Phys **56**, 62 (1988)

3 S Lee. Experiments with the ICTP-UM 3.3 kJ Plasma Fusion Facilities. ICTP-H4-SMR554/4 (1991)

4 S Lee. IEEE Trans on Plasma Sc, **19**, 912 (1991)

5 M H Liu, S Lee. SXR Radiation Modelling for Neon Plasma Focus. Intern Congress on Plasma Phys, Prague, June 1998

6 S Lee, P Lee, G Zhang, X Feng, V Gribkov, M Liu, A Serban, T K S Wong. IEEE Trans Plasma Sc, **26**, 1119 (1998)

Appendix

N A D Khattak has suggested improvements to the radiative phase.These suggestions have not yet been implemented in the code. The suggestions are recorded in a separate file called Appemdix to the Radiative Model.